ON THE DERIVATION OF A THEORY OF BENDING OF LAYERED PLATES

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M.I. GUSEIN-ZADE
(Moscow)

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The construction of a theory of bending of thin layered plates is described herein. The plates consist of layers having significantly different elastic properties. The investigation is carried out by a method of asymptotic integration of the three-dimensional equations of the theory of elasticity [1 and 2] in the narrow region occupied by the plate.

The complete state of stress in a layered plate consists of an internal state of stress and a state of stress corresponding to edge effects. The internal state of stress is investigated in this paper.

For layered plates consisting of alternating soft and stiff layers with elastic moduli E_1 and E_2, respectively, a classification of the problems will be made according to the magnitude of the ratio E_1/E_2. It is shown ~~that~~ in a wide range of variation of E_1/E_2 covering almost all possible cases of layered plates the problem of deformation of the plate under the action of arbitrary surface loading reduces in each approximation to the usual equations: the problem of flexure reduces to a biharmonic equation, and the inplane problem to the equations of generalized plane stress for some anisotropic plate.

The asymptotic method of constructing a theory of layered plates permits a unified approach to the problem of justification and establishment of the limits of applicability of any of the hypotheses upon which the various theories of layered plates are based. Moreover, this approach makes it possible to determine the shearing and normal stresses on planes parallel to the surface of the plate in addition to the flexural stresses. If these stresses on planes parallel to the surface play a subordinate role in homogeneous plates, they can be of primary importance for plates consisting of layers having radically different elastic properties. The determination of these stresses can be very significant for solution of the problem of the strength of the bonding between the various layers.

1. We shall consider a layered plate consisting of alternate stiff and soft layers. We shall consider that the top and bottom layers of the plate are stiff layers. We make use of an orthogonal system of curvilinear coordinate α, β, γ in which the γ-axis is perpendicular to the plane of the plate. The $\alpha - \beta$ coordinate plane may either pass through any layer (soft or stiff) or else coincide with any plane of separation between layers. For definiteness, we shall assume that it passes through a soft layer. We shall consider that this layer consists of two layers having the same elastic properties. We begin the numbering of the layers from the $\alpha - \beta$ plane using negative numbers for the layers located below this plane. The soft layers will then have odd numbers and the stiff layers even ones.

We shall consider that the individual layers of the plate have different thicknesses, elastic moduli and Poisson's ratios, but that the elastic moduli of all the soft layers are approximately E_1 and the elastic moduli of all the stiff layers are approximately E_2.

We express the ratio E_1/E_2 as some power of the dimensionless thichness $\varepsilon = h/l$ (the plate thickness is $2h$ and l is a characteristic plan dimension of the plate), i.e., we set

$E_1/E_2 = \varepsilon^a$.

The range of values of a close to zero, which corresponds to a layered plate composed of layers having comparable elastic properties has been considered in [3]. We shall study the range of values $a > 0$. In this case, a layered plate is characterized by two small parameters, ε and $E_1/E_2 = \varepsilon^a$.

Without loss of generality we may assume, firstly, that a takes on rational values, i.e., $a = p/q$ (where p and q are integers), and, secondly, that q is not a very large number. These assumptions may always be satisfied since there exists a certain arbitrariness in the choice of the quantities ε and E_1/E_2.

The asymptotic integration of the equations of the theory of elasticity will be carried out for such values of a by using expansions in the parameter $\lambda = \varepsilon^{1/q}$. The transition from this parameter to the fundamental parameters of the layered plate is carried out by λ to integral powers.

2. The internal states of stress and strain for a homogeneous layer were investigated in [4], where expansions in the parameters ε were obtained for displacements and stresses. To obtain the internal states of stress and strain of an individual layer of a layered plate, it is necessary to carry through an asymptotic integration of the Navier equations using expansions in the parameter $\lambda = \varepsilon^{1/q}$. This is easily accomplished by analogy to [4].

Let $u_\alpha^{(j)}$, $u_\beta^{(j)}$, $u_\gamma^{(j)}$ be the components of the displacement vector of the j-th layer. We introduce the dimensionless quantities

$$v_\alpha^{(j)} = \frac{u_\alpha^{(j)}}{h}, \quad v_\beta^{(j)} = \frac{u_\beta^{(j)}}{h}, \quad v_\gamma^{(j)} = \frac{u_\gamma^{(j)}}{h}, \quad \xi = \frac{\alpha}{l}, \quad \eta = \frac{\beta}{l}, \quad \zeta = \frac{\gamma}{h}$$

The solution of the system of equations obtained from the Navier equations for the j-th layer after transforming to the dimensionless variables is constructed in the form

$$v_\alpha^{(j)} = \varepsilon^{\varkappa+1} \sum_{s=0} \lambda^s v_\alpha^{(js)} \quad (\alpha\beta), \qquad v_\gamma^{(j)} = \varepsilon^{\varkappa} \sum_{s=0} \lambda^s v_\gamma^{(js)} \tag{2.1}$$

Here and in what follows the first superscript indicates the layer number to which the quantity refers and the second superscript is related to the approximation number.

We obtain equations for $v_\alpha^{(js)}$, $v_\beta^{(js)}$, $v_\gamma^{(js)}$ which are easily integrated with respect to ζ; this leads to a solution of the form

$$v_\alpha^{(js)} = \sum_{k=0}^{r+1} \zeta^k v_{\alpha k}^{(js)} \quad (\alpha\beta) \qquad v_\gamma^{(js)} = \sum_{k=}^{r} \zeta^k v_{\gamma k}^{(js)} \tag{2.2}$$

where

$$r = \begin{cases} [s/q] \quad , & \text{if} \quad [s/q] \quad \text{is even} \\ [s/q] - 1, & \text{if} \quad [s/q] \quad \text{is odd} \end{cases} \tag{2.3}$$

The brackets refer to the integral part of s/q.

The quantities $v_{\alpha k}^{(js)}$, $v_{\beta k}^{(js)}$, $v_{\gamma k}^{(js)}$ are functions of ξ and η; they are related by the following differential Eqs.

$$v_{\alpha, k+2}^{(js)} = \frac{1}{(k+1)(k+2)} \left\{ -\frac{3-2\nu_j}{2(1-\nu_j)} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} (H_\beta v_{\alpha k}^{(j, s-2q)}) + \right.\right.\right.$$
$$\left.\left. + \frac{\partial}{\partial \eta} (H_\alpha v_{\beta k}^{(j, s-2q)}) \right] \right\} + \frac{1}{H_\beta} \frac{\partial}{\partial \eta} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} (H_\beta v_{\beta k}^{(j, s-2q)}) - \frac{\partial}{\partial \eta} (H_\alpha v_{\alpha k}^{(j, s-2q)}) \right] \right\} +$$
$$\left. + \frac{1}{2(1-\nu_j)} \frac{1}{k} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \nabla v_{\gamma, k-1}^{(j, s-2q)} \right\} \quad (\alpha\beta) \tag{2.4}$$

for $k \geq 0$

$$v_{\gamma, k+2}^{(j, s)} = \frac{1}{(k+1)(k+2)}\left\{-\frac{1}{2(1-\nu_j)}(k+1)\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\xi}(H_\beta v_{\alpha, k+1}^{(j, s-2q)}) + \right.\right.$$
$$\left.\left. + \frac{\partial}{\partial\eta}(H_\alpha v_{\beta, k+1}^{(j, s-2q)})\right] - \frac{1-2\nu_j}{2(1-\nu_j)}\nabla v_{\gamma k}^{(j, s-2q)}\right\}$$

In addition, we find that

$$v_{\alpha 2}^{(js)} = \frac{1-\nu_j}{1-2\nu_j}\frac{1}{H_\alpha}\frac{\partial}{\partial\xi}\left\{\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\xi}(H_\beta v_{\alpha 0}^{(j, s-2q)}) + \frac{\partial}{\partial\eta}(H_\alpha v_{\beta 0}^{(j, s-2q)})\right]\right\} +$$
$$+ \frac{1}{2}\frac{1}{H_\beta}\frac{\partial}{\partial\eta}\left\{\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\xi}(H_\beta v_{\beta 0}^{(j, s-2q)}) - \frac{\partial}{\partial\eta}(H_\alpha v_{\alpha 0}^{(j, s-2q)})\right]\right\} -$$
$$- \frac{1}{2(1-2\nu_j)}\frac{1}{H_\alpha}\frac{\partial v_{\gamma 1}^{(js)}}{\partial\xi} \qquad (\alpha\beta) \tag{2.5}$$

where H_α and H_β are the reciprocals of the scale factors (i.e., the square roots of the metric coefficients). Eqs. (2.4) are recurrence relations which permit the determination of $v_{\alpha k}^{(js)}$, $v_{\beta k}^{(js)}$ (for $k \geq 3$), and $v_{\gamma k}^{(js)}$ (for $k \geq 2$) in terms of the quantities in the $(s - 2q)$-th approximation.

We now determine the stresses corresponding to the displacements of Eqs. (2.1). If we express the strain components in Hooke's Law in terms of displacements, transform to dimensionless quantities, and then substitute (2.1) for the displacements, we obtain

$$\frac{1}{E_j}\sigma_{\alpha\alpha}^{(j)} = \varepsilon^{\varkappa+2}\sum_{s=0}\lambda^s\sigma_{\alpha\alpha}^{(js)} \quad (\alpha\beta), \qquad \frac{1}{E_j}\sigma_{\alpha\beta}^{(j)} = \varepsilon^{\varkappa+2}\sum_{s=0}\lambda^s\sigma_{\alpha\beta}^{(js)} \tag{2.6}$$

$$\frac{1}{E_j}\sigma_{\alpha\gamma}^{(j)} = \varepsilon^{\varkappa+1}\sum_{s=0}\lambda^s\sigma_{\alpha\gamma}^{(js)} \quad (\alpha\beta), \qquad \frac{1}{E_j}\sigma_{\gamma\gamma}^{(j)} = \varepsilon^{\varkappa+2}\sum_{s=0}\lambda^s\sigma_{\gamma\gamma}^{(js)} \tag{2.7}$$

The quantities in Eqs. (2.6) and (2.7) with superscript s are polynomials in ζ

$$\sigma_{\alpha\alpha}^{(js)} = \sum_{k=0}^{r+1}\zeta^k\sigma_{\alpha\alpha k}^{(js)} \quad (\alpha\beta), \qquad \sigma_{\alpha\beta}^{(js)} = \sum_{k=0}^{r+1}\zeta^k\sigma_{\alpha\beta k}^{(js)} \tag{2.8}$$

$$\sigma_{\alpha\gamma}^{(js)} = \sum_{k=0}^{r}\zeta^k\sigma_{\alpha\beta k}^{(js)} \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(js)} = \sum_{k=0}^{r+1}\zeta^k\sigma_{\gamma\gamma k}^{(js)} \tag{2.9}$$

The value of r is determined in accordance with (2.3). The quantities with the subscript k in Eqs. (2.8) and (2.9) are functions of ξ and η which are related by the following Expressions:

$$\sigma_{\alpha\alpha k}^{(js)} = \frac{\nu_j}{(1+\nu_j)(1-2\nu_j)}\left\{\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\xi}(H_\beta v_{\alpha k}^{(js)}) + \frac{\partial}{\partial\eta}(H_\alpha v_{\beta k}^{(js)})\right] + \right.$$
$$\left. + (k+1)v_{\gamma, k+1}^{(j, s+2q)}\right\} + \frac{1}{1+\nu_j}\left[\frac{1}{H_\alpha}\frac{\partial v_{\alpha k}^{(js)}}{\partial\xi} - \frac{1}{H_\alpha H_\beta}\frac{\partial H_\alpha}{\partial\eta}v_{\beta k}^{(js)}\right] \quad (\alpha\beta) \tag{2.10}$$

$$\sigma_{\alpha\beta k}^{(js)} = \frac{1}{2(1+\nu_j)}\left[\frac{H_\beta}{H_\alpha}\frac{\partial}{\partial\xi}\left(\frac{v_{\beta k}^{(js)}}{H_\beta}\right) + \frac{H_\alpha}{H_\beta}\frac{\partial}{\partial\eta}\left(\frac{v_{\alpha k}^{(js)}}{H_\alpha}\right)\right]$$

$$\sigma_{\alpha\gamma k}^{(js)} = \frac{1}{2(1+\nu_j)}\left[\frac{1}{H_\alpha}\frac{\partial v_{\gamma k}^{(js)}}{\partial \xi} + (k+1)\, v_{\alpha, k+1}^{(js)}\right] \quad (\alpha\beta) \tag{2.11}$$

$$\sigma_{\gamma\gamma k}^{(js)} = \frac{\nu_j}{(1+\nu_j)(1-2\nu_j)}\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial \xi}(H_\beta v_{\alpha k}^{(js)}) + \frac{\partial}{\partial \eta}(H_\alpha v_{\beta k}^{(js)})\right] + \\ + \frac{1-\nu_j}{(1+\nu_j)(1-2\nu_j)}(k+1)\, v_{\gamma, k+1}^{(j, s+2q)}$$

The quantities

$$\sigma_{\alpha\alpha k}^{(js)} \ (\alpha\beta), \qquad \sigma_{\alpha\beta k}^{(js)}, \qquad \sigma_{\gamma\gamma k}^{(js)}, \qquad \sigma_{\alpha\gamma k}^{(js)} \ (\alpha\beta)$$

satisfy Eqs.

$$\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial \xi}(H_\beta \sigma_{\alpha\alpha k}^{(js)}) + \frac{\partial}{\partial \eta}(H_\alpha \sigma_{\alpha\beta k}^{(js)}) - \frac{\partial H_\beta}{\partial \xi}\sigma_{\beta\beta k}^{(js)} + \frac{\partial H_\alpha}{\partial \eta}\sigma_{\alpha\beta k}^{(js)}\right] + \\ + (k+1)\,\sigma_{\alpha\gamma, k+1}^{(j, s+2q)} = 0 \quad (\alpha\beta) \tag{2.12}$$

$$\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial \xi}(H_\beta \sigma_{\alpha\gamma k}^{(js)}) + \frac{\partial}{\partial \eta}(H_\alpha \sigma_{\beta\gamma k}^{(js)})\right] + (k+1)\,\sigma_{\gamma\gamma, k+1}^{(j, s)} = 0$$

3. We shall consider a layered plate for which $0 < a < 2$, acted upon by arbitrary surface loading. The boundary conditions have the form

$$\begin{aligned} \sigma_{\alpha\gamma}^{(2n)} &= \tau_\alpha^+(\alpha, \beta) \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(2n)} = \tau_\gamma^+(\alpha\beta) \quad \text{for} \quad \zeta = \zeta_{2n} \\ \sigma_{\alpha\gamma}^{(-2m)} &= \tau_\alpha^-(\alpha, \beta) \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(-2m)} = \tau_\gamma^-(\alpha\beta) \quad \text{for} \quad \zeta = \zeta_{-2m} \end{aligned} \tag{3.1}$$

Geometrical and statical conditions of bonding of the layers must be satisfied on the planes of contact between layers. We now write out these conditions for the layers $2k$ and $2k+1$, i.e., the condition for $\zeta = \zeta_{2k}$:

$$\begin{aligned} v_\alpha^{(2k)} &= v_\alpha^{(2k+1)} \quad (\alpha\beta), \qquad v_\gamma^{(2k)} = v_\gamma^{(2k+1)} \\ \sigma_{\alpha\gamma}^{(2k)} &= \sigma_{\alpha\gamma}^{(2k+1)} \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(2k)} = \sigma_{\gamma\gamma}^{(2k+1)} \end{aligned} \tag{3.2}$$

We denote the indices of the first nonzero terms in the expansions (2.7) by s_0 and s_* for the stiff and soft layers, respectively.

Taking into account (2.1) and (2.7), we obtain the boundary conditions and the conditions of bonding of the layers from (3.1) and (3.2). For $\zeta = \zeta_{2n}$ we have:

for the zeroth approximation

$$E_2 e_{2n}\varepsilon^{\varkappa+1}\lambda^{s_0}\sigma_{\alpha\gamma}^{(2n, s_0)} = \tau_\alpha^+(\xi, \eta) \quad (\alpha\beta), \qquad E_2 e_{2n}\varepsilon^{\varkappa+2}\lambda^{s_0}\sigma_{\gamma\gamma}^{(2n, s_0)} = \tau_\gamma^+(\xi, \eta) \tag{3.3}$$

for the s-th approximation

$$\sigma_{\alpha\gamma}^{(2n, s_0+s)} = 0 \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(2n, s_0+s)} = 0 \tag{3.4}$$

The boundary conditions for ζ_{-2m} have analogous forms for the various approximations. We now write out the conditions of bonding on the surfaces of the $(2k+1)$-th (soft) layer ($k = 0, 1, 2, \ldots, n-1$):

for $\zeta = \zeta_{2k+1}$

$$v_\alpha^{(2k+2, s)} = v_\alpha^{(2k+1, s)} \quad (\alpha\beta), \qquad v_\gamma^{(2k+2, s)} = v_\gamma^{(2k+1, s)} \tag{3.5}$$

$$\begin{aligned} E_2 e_{2k+2}\varepsilon^{\varkappa+1}\lambda^{s_0+s}\sigma_{\alpha\gamma}^{(2k+2, s_0+s)} &= E_1 e_{2k+1}\varepsilon^{\varkappa+1}\lambda^{s_*+s}\sigma_{\alpha\gamma}^{(2k+1, s_*+s)} \quad (\alpha\beta) \\ E_2 e_{2k+2}\varepsilon^{\varkappa+2}\lambda^{s_0+s}\sigma_{\gamma\gamma}^{(2k+2, s_0+s)} &= E_1 e_{2k+1}\varepsilon^{\varkappa+2}\lambda^{s_*+s}\sigma_{\gamma\gamma}^{(2k+1, s_*+s)} \end{aligned} \tag{3.6}$$

for $\zeta = \zeta_{2k}$

$$v_\alpha{}^{(2k,s)} = v_\alpha{}^{(2k+1,s)} \quad (\alpha\beta), \qquad v_\gamma{}^{(2k,s)} = v_\gamma{}^{(2k+1,s)} \tag{3.7}$$

$$\begin{gathered} E_2 e_{2k} \varepsilon^{\varkappa+1} \lambda^{s_0+s} \sigma_{\alpha\gamma}{}^{(2k,s_0+s)} = E_1 e_{2k+1} \varepsilon^{\varkappa+1} \lambda^{s_*+s} \sigma_{\alpha\gamma}{}^{(2k+1,s_*+s)} \quad (\alpha\beta) \\ E_2 e_{2k} \varepsilon^{\varkappa+2} \lambda^{s_0+s} \sigma_{\gamma\gamma}{}^{(2k,s_0+s)} = E_1 e_{2k+1} \varepsilon^{\varkappa+2} \lambda^{s_*+s} \sigma_{\gamma\gamma}{}^{(2k+1,s_*+s)} \end{gathered} \tag{3.8}$$

In Eqs. (3.3) to (3.8) we have taken $e_{2k} = E_{2k} / E_2$, $e_{2k+1} = E_{2k+1} / E_1$. It follows from what has been said earlier that e_{2k} and e_{2k+1} are not far from unity.

We shall now ascertain the possibility of satisfying the boundary conditions (3.3) and the statical conditions of bonding of the layers (3.6) and (3.8). On the basis of arguments which are analogous to those given in Section 3 of [4] for a homogeneous plate, we arrive at the conclusion that $s_0 = 2q$ and $s_* = 2q - p$. This means that in the Expressions (2.7) for $\sigma_{\alpha\gamma}, \sigma_{\beta\gamma}$ and $\sigma_{\gamma\gamma}$ the first $2q$ terms are zero for a stiff layer, and the first $2q - p$ are zero for a soft layer. From the condition that the indicated terms go to zero, we obtain the following relations for the j-th layer:

$$\frac{1}{H_\alpha} \frac{\partial v_{\gamma 0}^{(js)}}{\partial \xi} + v_{\alpha 1}^{(js)} = 0 \tag{3.9}$$

$$\frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta v_{\alpha 0}^{(js)} \right) + \frac{\partial}{\partial \eta} \left(H_\alpha v_{\beta 0}^{(js)} \right) \right] + \frac{1 - \nu_j}{\nu_j} v_{\gamma_1}{}^{(j,\, s+2q)} = 0 \tag{3.10}$$

If $j = 2k$ ($k = 1, 2, \ldots, n; -1, -2, \ldots, -m$) the conditions (3.9) and (3.10) are satisfied for the approximations $s = 0, 1, 2, \ldots, 2q - 1$.

If, however, $j = 2k - 1$ ($k = 1, 2, \ldots, n$) or $j = 2k + 1$ ($k = -1, -2, \ldots, -m$), then the conditions (3.9) and (3.10) are satisfied for $s = 0, 1, 2, \ldots, 2q - p - 1$.

The conditions (3.9) and (3.10) are equivalent to the satisfaction of the Kirchhoff assumption. Therefore, the Kirchhoff assumption is satisfied for the stiff layers in the first $2q$ approximations, and in the soft layers for the first $2q - p$ approximations. The first $2q$ approximations thus fall into two groups. The first group consists of the approximations $s = 0, 1, 2, \ldots, 2q - p - 1$. In these approximations the Kirchhoff assumption si satisfied for the whole layered plate as a unit. The approximations $s = 2q - p, 2q - p + 1, \ldots, 2q - 1$ form the second group. In these approximations the Kirchhoff assumption holds only in the stiff layers of the plate.

The character of the state of stress in the various layers of the plate also depends on the number of leading terms in the expansions (2.7) which go to zero. Let us estimate the orders of magnitude of the stresses present in the stiff and soft layers. Considering that $s_0 = 2q$ and $s_* = 2q - p$, we obtain from (2.6) and (2.7)

for the stiff layers

$$\sigma_{\alpha\alpha}^{(2k)},\ \sigma_{\beta\beta}^{(2k)},\ \sigma_{\alpha\beta}^{(2k)} \sim E_2 \varepsilon^{\varkappa+2}; \qquad \sigma_{\alpha\gamma}^{(2k)},\ \sigma_{\beta\gamma}^{(2k)} \sim E_2 \varepsilon^{\varkappa+3}; \qquad \sigma_{\gamma\gamma}^{(2k)} \sim E_2 \varepsilon^{\varkappa+4} \tag{3.11}$$

for the soft layers (3.12)

$$\sigma_{\alpha\alpha}^{(2k-1)},\ \sigma_{\beta\beta}^{(2k-1)},\ \sigma_{\alpha\beta}^{(2k-1)} \sim E_1 \varepsilon^{\varkappa+2}; \quad \sigma_{\alpha\gamma}^{(2k-1)},\ \sigma_{\beta\gamma}^{(2k-1)} \sim E_1 \varepsilon^{\varkappa+3-a}; \quad \sigma_{\gamma\gamma}^{(2k-1)} \sim E_1 \varepsilon^{\varkappa+4-a}$$

It follows from (3.11) that the stiff layers act essentially in flexure, since the flexural stresses $\sigma_{\alpha\alpha}^{(2k)}$, $\sigma_{\beta\beta}^{(2k)}$, and $\sigma_{\alpha\beta}^{(2k)}$ are the largest. The shearing stresses $\sigma_{\alpha\gamma}^{(2k)}$, and $\sigma_{\beta\gamma}^{(2k)}$ are one order smaller than the flexural stresses, and the normal stresses $\sigma_{\gamma\gamma}^{(2k)}$ are two orders smaller.

It follows from (3.12) that in the soft layers the significance of the shearing stresses $\sigma_{\alpha\gamma}^{(2k-1)}$, and $\sigma_{\beta\gamma}^{(2k-1)}$ of the normal stresses $\sigma_{\gamma\gamma}^{(2k-1)}$ becomes greater as a increases.

This indicates that as a gets larger, the soft layers act more and more in shear and compression. We remark that in the first p approximations the shearing stresses are constant through the thickness of the soft layers. Moreover, as can be seen from (3.12), the order of magnitude of these stresses for $1 < a < 2$ is greater than the order of magnitude of the flexural stresses. This means that for $1 < a < 2$, Reissner's assumption [5] is satisfied in the soft layers in the first p approximations.

4. The expressions for the displacements and stresses in any layer contain six arbitrary functions for each approximation. In the s-th approximation the following are arbitrary functions:

for a stiff layer

$$v_{\alpha 0}^{(2k,s)}, \quad v_{\beta 0}^{(2k,s)}, \quad v_{\gamma 0}^{(2k,s)}, \quad \sigma_{\alpha\gamma 0}^{(2k,2q+s)}, \quad \sigma_{\beta\gamma 0}^{(2k,2q+s)}, \quad \sigma_{\gamma\gamma 0}^{(2k,2q+s)}$$

for a soft layer

$$v_{\alpha 0}^{(2k-1,s)}, \quad v_{\beta 0}^{(2k-1,s)}, \quad v_{\gamma 0}^{(2k-1,s)}, \quad \sigma_{\alpha\gamma 0}^{(2k-1,2q-p+s)}, \quad \sigma_{\beta\gamma 0}^{(2k-1,2q-p+s)}, \quad \sigma_{\gamma\gamma 0}^{(2k-1,2q-p+s)}$$

If in the s-th approximation we estimate the six arbitrary functions corresponding to a soft layer from the twelve bonding conditions of this soft layer with the two adjacent stiff layers, (3.5) to (3.8), we then obtain six constraint conditions [4] between neighboring stiff layers in the s-th approximation. The constraint conditions between neighboring stiff layers make it possible to eliminate the soft layers from consideration. These relations contain the twelve arbitrary functions corresponding to the s-th approximation for the stiff layers adjacent to the soft layer in question. In addition, these conditions contain quantities relating to the previous approximations, which we consider as known when constructing a given approximation.

For the zeroth approximation the constraint relations for the layers $2k$ and $2k+2$ have the form

$$v_{\alpha 0}^{(2k,0)} = v_{\alpha 0}^{(2k+2,0)} \quad (\alpha\beta), \qquad v_{\gamma 0}^{(2k,0)} = v_{\gamma 0}^{(2k+2,0)} \tag{4.1}$$

$$e_{2k}\left[\sigma_{\alpha\gamma 0}^{(2k,2q)} + \zeta_{2k}\sigma_{\alpha\gamma 1}^{(2k,2q)} + \zeta_{2k}^{2}\sigma_{\alpha\gamma 2}^{(2k,2q)}\right] = e_{2k+2}\left[\sigma_{\alpha\gamma 0}^{(2k+2,2q)} + \zeta_{2k+1}\sigma_{\alpha\gamma 1}^{(2k+2,2q)} + \right.$$
$$\left. + \zeta_{2k+1}^{2}\sigma_{\alpha\gamma 2}^{(2k+2,2q)}\right] \quad (\alpha\beta) \tag{4.2}$$

$$e_{2k}\left[\sigma_{\gamma\gamma 0}^{(2k,2q)} - \zeta_{2k}^{2}\sigma_{\gamma\gamma 2}^{(2k,2q)} - 2\zeta_{2k}^{3}\sigma_{\gamma\gamma 3}^{(2k,2q)}\right] =$$
$$= e_{2k+2}\left[\sigma_{\gamma\gamma 0}^{(2k+2,2q)} - \zeta_{2k+1}^{2}\sigma_{\gamma\gamma 2}^{(2k+2,2q)} - 2\zeta_{2k+1}^{3}\sigma_{\gamma\gamma 3}^{(2k+2,2q)}\right]$$

For a layered plate consisting of $n + m$ stiff layers, there are $6(n + m - 1)$ constraint conditions for the stiff layers and six boundary conditions for $\zeta = \zeta_{2n}$ and $\zeta = \zeta_{-2m}$ for each approximation. In these $6(n + m)$ equations, $6(n + m)$ arbitrary functions occur which refer to the stiff layers and correspond to the approximation in question.

5. We now show that the problem of constructing any approximation reduces to the solution of three two-dimensional differential equations.

We first observe that the values of some of the quantities are closely connected with this approximation, while the values of others are determined from the previous approximations. There are three classes of quantities in the latter group: some are expressed in terms of the $(s - 2q)$-th approximation, others in terms of the $(s - 2q + p)$-th approximation, and the third set in terms of the $(s - 2q)$-th and $(s - 2q + p)$-th approximations jointly. The superscript s which indicates the number of the approximation will not be used to denote these quantities. Instead, we shall use the supercripts θ, $\varkappa$, and r for the three types. Moreover, we shall agree to drop the superscript indicating layer number on quantities which refer to the layered plate as a whole.

From the geometric constraint conditions between stidd layers, it follows that

$$v_{\alpha 0}^{(2k,\, s)} = v_{\alpha 0}^{(s)} \quad (\alpha\ \beta), \quad v_{\gamma 0}^{(2k,\, s)} = v_{\gamma 0}^{(s)} \qquad (0 \leqslant s \leqslant 2q - p - 1) \tag{5.1}$$

$$v_{\alpha 0}^{(2k,\, s)} = v_{\alpha 0}^{(s)} + v_{\alpha 0}^{(2k,\, \varkappa)} \quad (\alpha\ \beta), \quad v_{\gamma 0}^{(2k,\, s)} = v_{\gamma 0}^{(s)} \qquad (2q - p \leqslant s \leqslant 2q - 1) \tag{5.2}$$

$$v_{\alpha 0}^{(2k,\, s)} = v_{\alpha 0}^{(s)} + v_{\alpha 0}^{(2k,\, \theta)} \quad (\alpha\ \beta), \quad v_{\gamma 0}^{(2k,\, s)} = v_{\gamma 0}^{(s)} + v_{\gamma 0}^{(2k,\, \theta)} \qquad (2q \leqslant s \leqslant 4q - p - 1) \tag{5.3}$$

$$(k = 1, 2, \ldots, n; -1, -2, \ldots, -m)$$

In (5.2) and (5.3) we take

$$v_{\alpha 0}^{(s)} = v_{\alpha 0}^{(2n,\, s)} \quad (\alpha\ \beta), \qquad v_{\gamma 0}^{(s)} = v_{\gamma 0}^{(2n,\, s)}$$

Then we have

for the approximations $2q - p \leqslant s \leqslant 2q - 1$

$$v_{\alpha 0}^{(2n,\, \varkappa)} = 0 \qquad (\alpha\ \beta) \tag{5.4}$$

$$v_{\alpha 0}^{(2n-2,\, \varkappa)} = -2(1 + \nu_{2n-1})(\zeta_{2n-1} - \zeta_{2n-2})\,\sigma_{\alpha\gamma 0}^{(2n-1,\, s)} \qquad (\alpha\ \beta)$$

$$v_{\alpha 0}^{(2n-4,\, \varkappa)} = v_{\alpha 0}^{(2n-2,\, \varkappa)} - 2(1 + \nu_{2n-3})(\zeta_{2n-3} - \zeta_{2n-4})\,\sigma_{\alpha\gamma 0}^{(2n-3,\, s)} \qquad (\alpha\ \beta)$$

for the approximations $2q \leqslant s \leqslant 4q - p - 1$

$$v_{\alpha 0}^{(2n,\, \theta)} = 0 \quad (\alpha\ \beta), \qquad v_{\alpha 0}^{(2n-2,\, \theta)} = V_{\alpha}^{(2n-2,\, \theta)} \quad (\alpha\ \beta)$$

$$v_{\alpha 0}^{(2n-4,\, \theta)} = v_{\alpha 0}^{(2n-2,\, \theta)} + V_{\alpha}^{(2n-4,\, \theta)} \quad (\alpha\ \beta), \ldots \tag{5.5}$$

$$v_{\gamma 0}^{(2n,\, \theta)} = 0, \quad v_{\gamma 0}^{(2n-2,\, \theta)} = V_{\gamma}^{(2n-2,\, \theta)}, \quad v_{\gamma 0}^{(2n-4,\, \theta)} = v_{\gamma 0}^{(2n-2,\, \theta)} + V_{\gamma}^{(2n-4,\, \theta)}, \ldots \tag{5.6}$$

where (5.7)

$$\begin{aligned} V_{\alpha}^{(2k,\, \theta)} = {} & (\zeta_{2k+1} v_{\alpha 1}^{(2k+2,\, \theta)} - \zeta_{2k} v_{\alpha 1}^{(2k,\, \theta)}) - (\zeta_{2k+1} - \zeta_{2k})\, v_{\alpha 1}^{(2k+1,\, \theta)} + \\ & + (\zeta_{2k+1}^2 v_{\alpha 2}^{(2k+2,\, \theta)} - \zeta_{2k}^2 v_{\alpha 2}^{(2k,\, \theta)}) - (\zeta_{2k+1}^2 - \zeta_{2k}^2)\, v_{\alpha 2}^{(2k+1,\, \theta)} + \\ & + (\zeta_{2k+1}^3 v_{\alpha 3}^{(2k+2,\, \theta)} - \zeta_{2k}^3 v_{\alpha 3}^{(2k,\, \theta)}) - (\zeta_{2k+1}^3 - \zeta_{2k}^3)\, v_{\alpha 3}^{(2k+1,\, \theta)} \qquad (\alpha\beta) \end{aligned}$$

$$V_{\gamma}^{(2k,\, \theta)} = \zeta_{2k+1} v_{\gamma 1}^{(2k+2,\, \theta)} - \zeta_{2k} v_{\gamma 1}^{(2k,\, \theta)} - (\zeta_{2k+1} - \zeta_{2k})\, v_{\gamma 1}^{(2k+1,\, \theta)}$$

For the various approximations, if the statical constraint conditions are written out successively for each pair of neighboring stiff layers, if these are then added and the boundary conditions taken into account, we obtain as a result the following basic relations:

for the approximations $s = 0$

$$P_{\alpha\gamma 1}^{(0)} + P_{\alpha\gamma 2}^{(0)} = \frac{1}{E_2 \varepsilon^{\varkappa+3}} (\tau_{\alpha}^{+} - \tau_{\alpha}^{-}) \qquad (\alpha\beta)$$

$$P_{\gamma\gamma 2}^{(0)} + 2P_{\gamma\gamma 3}^{(0)} = -\frac{1}{E_2 \varepsilon^{\varkappa+4}} \left\{ (\tau_{\gamma}^{+} - \tau_{\gamma}^{-}) + \right. \tag{5.8}$$

$$\left. + \varepsilon \frac{1}{H_{\alpha} H_{\beta}} \left[\frac{\partial H_{\beta} (\zeta_{2n} \tau_{\alpha}^{+} - \zeta_{-2m} \tau_{\alpha}^{-})}{\partial \xi} + \frac{\partial H_{\alpha} (\zeta_{2n} \tau_{\beta}^{+} - \zeta_{-2m} \tau_{\beta}^{-})}{\partial \eta} \right] \right\}$$

for the approximations $1 \leqslant s \leqslant 2q - p - 1$

$$P_{\alpha\gamma 1}^{(s)} + P_{\alpha\gamma 2}^{(s)} = 0 \quad (\alpha\beta) \qquad P_{\gamma\gamma 2}^{(s)} + 2P_{\gamma\gamma 3}^{(s)} = 0 \tag{5.9}$$

for the approximations $2q - p \leqslant s \leqslant 2q - 1$

$$P_{\alpha\gamma 1}^{(s)} + P_{\alpha\gamma 2}^{(s)} = - R_{\alpha\gamma 1}^{(s-p)} - R_{\alpha\gamma 2}^{(s-p)} \quad (\alpha\beta)$$
$$P_{\gamma\gamma 2}^{(s)} + 2 P_{\gamma\gamma 3}^{(s)} = - R_{\gamma\gamma 2}^{(s-p)} - 2 R_{\gamma\gamma 3}^{(s-p)} \tag{5.10}$$

for the approximations $2q \leqslant s \leqslant 4q - p - 1$

$$P_{\alpha\gamma 1}^{(s)} + P_{\alpha\gamma 2}^{(s)} = - P_{\alpha\gamma 3}^{(s)} - P_{\alpha\gamma 4}^{(s)} - R_{\alpha\gamma 1}^{(s-p)} - R_{\alpha\gamma 2}^{(s-p)} \quad (\alpha\beta)$$
$$P_{\gamma\gamma 2}^{(s)} + 2 P_{\gamma\gamma 3}^{(s)} = - 3 P_{\gamma\gamma 4}^{(s)} - 4 P_{\gamma\gamma 5}^{(s)} - R_{\gamma\gamma 2}^{(s-p)} - 2 R_{\gamma\gamma 3}^{(s-p)} \tag{5.11}$$

for the approximations $4q - p \leqslant s \leqslant 4q - 1$

$$P_{\alpha\gamma 1}^{(s)} + P_{\alpha\gamma 2}^{(s)} = - P_{\alpha\gamma 3}^{(s)} - P_{\alpha\gamma 4}^{(s)} - R_{\alpha\gamma 1}^{(s-p)} - R_{\alpha\gamma 2}^{(s-p)} - R_{\alpha\gamma 3}^{(s-p)} - R_{\alpha\gamma 4}^{(s-p)} \quad (\alpha\beta)$$
$$P_{\gamma\gamma 2}^{(s)} + 2 P_{\gamma\gamma 3}^{(s)} = - 3 P_{\gamma\gamma 4}^{(s)} - 4 P_{\gamma\gamma 5}^{(s)} - R_{\gamma\gamma 2}^{(s-p)} - 2 R_{\gamma\gamma 3}^{(s-p)} - 3 R_{\gamma\gamma 4}^{(s-p)} - 4 R_{\gamma\gamma 5}^{(s-p)} \tag{5.12}$$

where the following notation has been used

$$P_{\alpha\gamma i}^{(t)} = \sum\nolimits^{*} (\zeta_{2k}^{\ i} - \zeta_{2k-1}^{\ i}) e_{2k} \sigma_{\alpha\gamma i}^{(2k,\ t+2q)} \quad (\alpha\beta)$$
$$R_{\alpha\gamma i}^{(t)} = \sum\nolimits^{*} (\zeta_{2k-1}^{\ i} - \zeta_{2k-2}^{\ i}) e_{2k-1} \sigma_{\alpha\gamma i}^{(2k-1,\ t+2q)} \quad (\alpha\beta)$$
$$P_{\gamma\gamma i}^{(t)} = \sum\nolimits^{*} (\zeta_{2k}^{\ i} - \zeta_{2k-1}^{\ i}) e_{2k} \sigma_{\gamma\gamma i}^{(2k,\ t+2q)}$$
$$R_{\gamma\gamma i}^{(t)} = \sum\nolimits^{*} (\zeta_{2k-1}^{\ i} - \zeta_{2k-2}^{\ i}) e_{2k-1} \sigma_{\gamma\gamma i}^{(2k-1,\ t+2q)} \tag{5.13}$$

with

$$\sum\nolimits^{*} f(a_{2k},\ b_{2k-1},\ c_{2k-2}) = \sum_{k=1}^{n} f(a_{2k},\ b_{2k-1},\ c_{2k-2}) - \sum_{k=-1}^{-m} f(a_{2k},\ b_{2k+1},\ c_{2k+2}) \tag{5.14}$$

The quantities on the left-hand sides of the basic relations (5.8) to (5.12) have the following Expressions:

$$P_{\alpha\gamma 1}^{(s)} = - \sum\nolimits^{*} (\zeta_{2k} - \zeta_{2k-1}) e_{2k} L(v_{\alpha 0}^{(s)};\ v_{\beta 0}^{(s)};\ \nu_{2k}) + P_{\alpha\gamma 1}^{(\tau)} \quad (\alpha\beta)$$
$$P_{\alpha\gamma 2}^{(s)} = Q_2 \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \nabla v_{\gamma 0}^{(s)} + P_{\alpha\gamma 2}^{(\theta)} \quad (\alpha\beta)$$
$$P_{\gamma\gamma 2}^{(s)} = Q_2 \nabla \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} (H_\beta v_{\alpha 0}^{(s)}) + \frac{\partial}{\partial \eta} (H_\alpha v_{\beta 0}^{(s)}) \right] \right\} + P_{\gamma\gamma 2}^{(\tau)}$$
$$P_{\gamma\gamma 3}^{(s)} = - 1/2\, Q_3 \nabla\nabla v_{\gamma 0}^{(s)} + P_{\gamma\gamma 3}^{(\theta)} \tag{5.15}$$

where

$$L(v_{\alpha 0}^{(s)},\ v_{\beta 0}^{(s)};\ \nu_{2k}) = \frac{1}{1 - \nu_{2k}^2} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} (H_\beta v_{\alpha 0}^{(s)}) + \frac{\partial}{\partial \eta} (H_\alpha v_{\beta 0}^{(s)}) \right] \right\} -$$
$$- \frac{1}{2(1 + \nu_{2k})} \frac{1}{H_\beta} \frac{\partial}{\partial \eta} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} (H_\beta v_{\beta 0}^{(s)}) - \frac{\partial}{\partial \eta} (H_\alpha v_{\alpha 0}^{(s)}) \right] \right\} \quad (\alpha\beta) \tag{5.16}$$

$$Q_i = \frac{1}{i} \sum\nolimits^{*} (\xi_{2k}^{\ i} - \xi_{2k-1}^{\ i}) \frac{e_{2k}}{1 - \nu_{2k}^{\ 2}} \tag{5.17}$$

For the quantities with superscripts τ or θ in (5.15), we have

$$P_{\alpha\gamma 1}^{(\tau)} = -\sum^{*} (\zeta_{2k} - \zeta_{2k-1}) e_{2k} \left[L\left(v_{\alpha 0}^{(2k,\tau)},\ v_{\beta 0}^{(2k,\tau)};\ \nu_{2k}\right) + \frac{\nu_{2k}}{1-\nu_{2k}} \frac{1}{H_\alpha} \frac{\partial \sigma_{\gamma\gamma 0}^{(2k,s)}}{\partial \xi} \right] \quad (\alpha\beta)$$

$$P_{\alpha\gamma 2}^{(\theta)} = \frac{1}{2} \sum^{*} (\zeta_{2k}^2 - \zeta_{2k-1}^2) e_{2k} \left\{ \frac{1}{1-\nu_{2k}^2} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \nabla v_{\gamma 0}^{(2k,\theta)} - \right. \tag{5.18}$$

$$- \frac{2-\nu_{2k}}{1-\nu_{2k}} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta \sigma_{\alpha\gamma 0}^{(2k,s)}\right) + \frac{\partial}{\partial \eta} \left(H_\alpha \sigma_{\beta\gamma 0}^{(2k,s)}\right) \right] \right\} +$$

$$\left. + \frac{1}{H_\beta} \frac{\partial}{\partial \eta} \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta \sigma_{\beta\gamma 0}^{(2k,s)}\right) - \frac{\partial}{\partial \eta} \left(H_\alpha \sigma_{\alpha\gamma 0}^{(2k,s)}\right) \right] \right\} \right\} \quad (\alpha\beta)$$

$$P_{\gamma\gamma 2}^{(\tau)} = \frac{1}{2} \sum^{*} (\zeta_{2k}^2 - \zeta_{2k-1}^2) e_{2k} \left\{ \frac{1}{1-\nu_{2k}^2} \nabla \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta v_{\alpha 0}^{(2k,\tau)}\right) + \right. \right. \right.$$

$$\left. \left. \left. + \frac{\partial}{\partial \eta} \left(H_\alpha v_{\beta 0}^{(2k,\tau)}\right) \right] \right\} + \frac{\nu_{2k}}{1-\nu_{2k}} \nabla \sigma_{\gamma\gamma 0}^{(2k,s)} \right\}$$

$$P_{\gamma\gamma 3}^{(\theta)} = -\frac{1}{6} \sum^{*} (\zeta_{2k}^3 - \zeta_{2k-1}^3) e_{2k} \left\{ \frac{1}{1-\nu_{2k}^2} \nabla\nabla v_{\gamma 0}^{(2k,\theta)} - \right.$$

$$\left. - \frac{2-\nu_{2k}}{1-\nu_{2k}} \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta \sigma_{\alpha\gamma 0}^{(2k,s)}\right) + \frac{\partial}{\partial \eta} \left(H_\alpha \sigma_{\beta\gamma 0}^{(2k,s)}\right) \right] \right\}$$

It is not difficult to verify that the right-hand sides of (5.10) to (5.12) are expressed in terms of quantities which are known from the preceding approximations. For example, the quantities on the right in Eqs. (5.10) have the form

$$R_{\alpha\gamma 1}^{(s-p)} = -\sum^{*} (\zeta_{2k-1} - \zeta_{2k-2}) e_{2k-1} \left[L\left(v_{\alpha 0}^{(2k-1,\,s-p)},\ v_{\beta 0}^{(2k-1,\,s-p)};\ \nu_{2k-1}\right) + \right.$$

$$\left. + \frac{\nu_{2k-1}}{1-\nu_{2k-1}} \frac{1}{H_\alpha} \frac{\partial \sigma_{\gamma\gamma 0}^{(2k-1,\,s-p)}}{\partial \xi} \right] \quad (\alpha\beta)$$

$$R_{\alpha\gamma 2}^{(s-p)} = Q_3^{*} \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \nabla v_{\gamma 0}^{(s-p)} \quad (\alpha\beta) \tag{5.19}$$

$$R_{\gamma\gamma 2}^{(s-p)} = Q_3^{*} \nabla \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta v_{\alpha 0}^{(s-p)}\right) + \frac{\partial}{\partial \eta} \left(H_\alpha v_{\beta 0}^{(s-p)}\right) \right] \right\}$$

$$R_{\gamma\gamma 3}^{(s-p)} = -\frac{1}{2} Q_3^{*} \nabla\nabla v_{\gamma 0}^{(s-p)}$$

where

$$Q_i^{*} = \frac{1}{i} \sum^{*} (\zeta_{2k-1}^{i} - \zeta_{2k-2}^{i}) \frac{e_{2k-1}}{1-\nu_{2k-1}^2} \tag{5.20}$$

Substituting (5.15) into the basic relations (5.8) to (5.12) and shifting to the right-hand sides those quantities which are expressed in terms of the preceding approximations, we obtain the following equations in $v_{\alpha 0}^{(s)}$, $v_{\beta 0}^{(s)}$, $v_{\gamma 0}^{(s)}$:

$$-\sum^{*} (\zeta_{2k} - \zeta_{2k-1}) e_{2k} L\left(v_{\alpha 0}^{(s)},\ v_{\beta 0}^{(s)};\ \nu_{2k}\right) + Q_3 \frac{1}{H_\alpha} \frac{\partial}{\partial \xi} \nabla v_{\gamma 0}^{(s)} = T_\alpha^{(s)} \quad (\alpha\beta) \tag{5.21}$$

$$Q_3 \nabla \left\{ \frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \xi} \left(H_\beta v_{\alpha 0}^{(s)}\right) + \frac{\partial}{\partial \eta} \left(H_\alpha v_{\beta 0}^{(s)}\right) \right] \right\} - Q_3 \nabla\nabla v_{\gamma 0}^{(s)} = T_\gamma^{(s)} \tag{5.22}$$

The right-hand sides, $T_\alpha^{(s)}$, $T_\beta^{(s)}$, $T_\gamma^{(s)}$ are expressed in terms of quantities which are known from the previous approximations.

For arbitrary location of the $\alpha - \beta$ coordinate plane, all three unknowns, $v_{\alpha 0}^{(s)}$, $v_{\beta 0}^{(s)}$, and $v_{\gamma 0}^{(s)}$, occur in each of Eqs. (5.21) and (5.22). However, the $\alpha - \beta$ coordinate plane can always be located so that the quantity Q_2 goes to zero*. Then Eqs. (5.21) will contain only the unknowns $v_{\alpha 0}^{(s)}$, and $v_{\beta 0}^{(s)}$, and Eq. (5.22) will contain only $v_{\gamma 0}^{(s)}$. This means that for such a choice of location of the $\alpha - \beta$ coordinate plane the generalized plane stress problem and the bending problem uncouple in each approximation. In what follows it will always be assumed that the $\alpha - \beta$ coordinate plane is located so that $Q_2 = 0$. It then follows from Eqs. (5.21) and (5.22) that $v_{\alpha 0}^{(s)}$, and $v_{\beta 0}^{(s)}$ satisfy Eqs.

$$-\sum{}^{*} (\zeta_{2k} - \zeta_{2k-1})\, e_{2k} L\left(v_{\alpha 0}^{(s)}, v_{\beta 0}^{(s)}; \nu_{2k}\right) = T_\alpha^{(s)} \quad (\alpha\beta) \tag{5.23}$$

and that $v_{\gamma 0}^{(s)}$ satisfies Eq.

$$-Q_3 \nabla\nabla v_{\gamma 0}^{(s)} = T_\gamma^{(s)} \tag{5.24}$$

In each approximation the determination of $v_{\alpha 0}^{(s)}$, and $v_{\beta 0}^{(s)}$ reduces to the solution of the two Eqs. of (5.23), which are the equations of generalized plane stress for some anisotropic plate. Under the additional condition that the Poisson's ratios of all the stiff layers are the same, i.e., under the condition $\nu_{2k} = \nu_2$, we obtain the equations of generalized plane stress for some isotropic plate from Eqs. (5.23)

$$-(1 - \nu_2^2)\, Q_1 L\left(v_{\alpha 0}^{(s)}, v_{\beta 0}^{(s)}; \nu\right) = T_\alpha^{(s)} \quad (\alpha\beta) \tag{5.25}$$

The quantity Q_1 is determined in accordance with (5.17) with $\nu_{2k} = \nu_2$.

In each approximation the problem of bending reduces to the solution of the biharmonic Eq. (5.24). The plate stiffness is found in terms of Q_3, which, as is clear from (5.17), depends on quantities referring to the stiff layers. Therefore, for $0 < a < 2$ the stiffness of a layered plate does not depend on the stiffness of the soft layers.

The Eqs. (5.23) to (5.25) differ for the various approximations only by their right-hand sides. Changes in the character of the right-hand sides occur in going from the approximations $s = 2q - p - 1$ to $s = 2q - p$, from $s = 2q - 1$ to $s = 2q$, from $s = 4q - p - 1$ to $s = 4q - p$, and from $s = 4q - 1$ to $s = 4q$, etc.

For the zeroth approximation the right-hand sides of Eqs. (5.19) and (5.20) have the form

$$T_\alpha^{(0)} = \frac{1}{E_2 \varepsilon^{\varkappa+3}} (\tau_\alpha^{+} - \tau_\alpha^{-}) \quad (\alpha\beta) \tag{5.26}$$

$$T_\gamma^{(0)} = -\frac{1}{E_2 \varepsilon^{\varkappa+4}} \left\{ \tau_\gamma^{+} - \tau_\gamma^{-} + \varepsilon \frac{1}{H_\alpha H_\beta} \left[\frac{\partial H_\beta (\tau_\alpha^{+} \zeta_{2n} - \tau_\alpha^{-} \zeta_{-2m})}{\partial \xi} - \frac{\partial H_\alpha (\tau_\beta^{+} \zeta_{2n} - \tau_\beta^{-} \zeta_{-2m})}{\partial \eta} \right] \right\}$$

For the approximations $1 \leqslant s \leqslant 2q - p - 1$, the right-hand sides of Eqs. (5.23) to (5.25) become zero

$$T_\alpha^{(s)} = 0 \quad (\alpha\beta), \qquad T_\gamma^{(s)} = 0 \tag{5.27}$$

For the approximation $2q - p \leqslant s \leqslant 2q - 1$, we have

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* It can happen that the quantity Q_2 becomes zero when the $\alpha - \beta$ plane either passes through a stiff layer or coincides with some plane of contact between layers. All the equations which have been given must then be altered somewhat.

$$T_\alpha^{(s)} = - R_{\alpha\gamma 1}^{(s-p)} - R_{\alpha\gamma 2}^{(s-p)} - P_{\alpha\gamma 1}^{(\tau)} - P_{\alpha\gamma 2}^{(0)} \quad (\alpha 3)$$
$$T_\gamma^{(s)} = - R_{\gamma\gamma 2}^{(s-p)} - 2 R_{\gamma\gamma 3}^{(s-p)} - P_{\gamma\gamma 2}^{(\tau)} - 2 P_{\gamma\gamma 3}^{(0)} \tag{5.28}$$

The expressions for the quantities on the right side of (5.28) are given in (5.18) and (5.19).

For the approximations $2q \leqslant s \leqslant 4q - p$, we obtain

$$T_\alpha^{(s)} = - P_{\alpha\gamma 3}^{(s)} - P_{\alpha\gamma 4}^{(s)} - R_{\alpha\gamma 1}^{(s-p)} - R_{\alpha\gamma 2}^{(s-p)} - P_{\alpha\gamma 1}^{(\tau)} - P_{\alpha\gamma 2}^{(0)} \quad (\alpha\beta)$$
$$T_\gamma^{(s)} = - 3 P_{\gamma\gamma 4}^{(s)} - 4 P_{\gamma\gamma 5}^{(s)} - R_{\gamma\gamma 2}^{(s-p)} - 2 R_{\gamma\gamma 3}^{(s-p)} - P_{\gamma\gamma 2}^{(\tau)} - 2 P_{\gamma\gamma 3}^{(0)} \tag{5.29}$$

6. Let us consider a layered plate for which the ratio E_1/E_2 is comparable with the square of the relative thickness, i.e., $a \sim 2$. In this case we shall use expansions in the parameter ε for the asymptotic integration of the equations of the theory of elasticity. In ascertaining the possibility of satisfying the boundary conditions and the conditions of bonding of the layers, we verify that in the expansions for the stresses $\sigma_{\alpha\gamma}$, $\sigma_{\beta\gamma}$, $\sigma_{\gamma\gamma}$, all terms are retained for the soft layers, but the first two terms are zero for the stiff layers. This is equivalent to saying that in the first two approximations (for expansions in ε), the Kirchhoff assumption holds only for the stiff layers.

The character of the state of stress in the soft layers is described in detail in [4] (see Section 5, State of Stress C).

In the zeroth approximation the constraint conditions for neighboring stiff layers have the form

$$e_{2k+1}\left[v_{\alpha 0}^{(2k+2,\,0)} - v_{\alpha 0}^{(2k,\,0)}\right] = 2\, e_{2k+2}\,(1+\nu_{2k+1})\,(\zeta_{2k+1}-\zeta_{2k})\left[\sigma_{\alpha\gamma 0}^{(2k+2,\,2)} + \zeta_{2k+1}\sigma_{\alpha\gamma 1}^{(2k+2,\,2)} + \zeta_{2k+1}^{2}\sigma_{\alpha\gamma 2}^{(2k+2,\,2)}\right] \quad (\alpha 3)$$
$$e_{2k+1}\left[v_{\alpha 0}^{(2k+2,\,0)} - v_{\alpha 0}^{(2k,\,0)}\right] = 2\, e_{2k}\,(1+\nu_{2k+1})\,(\zeta_{2k+1}-\zeta_{2k})\left[\sigma_{\alpha\gamma 0}^{(2k,\,2)} + \zeta_{2k}\sigma_{\alpha\gamma 1}^{(2k,\,2)} + \zeta_{2k}^{2}\sigma_{\alpha\gamma 2}^{(2k,\,2)}\right] \quad (\alpha\beta) \tag{6.1}$$

$$v_{\gamma 0}^{(2k,\,0)} = v_{\gamma 0}^{(2k+2,\,0)} \tag{6.2}$$

$$e_{2k}\left[\sigma_{\gamma\gamma 0}^{(2k,\,2)} - \zeta_{2k}^{2}\sigma_{\gamma\gamma 2}^{(2k,\,2)} - 2\,\zeta_{2k}^{3}\sigma_{\gamma\gamma 3}^{(2k,\,2)}\right] = e_{2k+2}\left[\sigma_{\gamma\gamma 0}^{(2k+2,\,2)} - \zeta_{2k+1}^{2}\sigma_{\gamma\gamma 2}^{(2k+2,\,2)} - 2\,\zeta_{2k+1}^{3}\sigma_{\gamma\gamma 3}^{(2k+2,\,2)}\right] \tag{6.3}$$

It follows from (6.2) that

$$v_{\gamma 0}^{(2k,\,0)} = v_{\gamma 0}^{(0)} \tag{6.4}$$

i.e., the quantity $v_{\gamma 0}^{(0)}$ is common for the entire layered plate. It is clear from (6.1) that the quantities $v_{\alpha 0}^{(2k,0)}$ and $v_{\beta 0}^{(2k,0)}$ are different for different layers.

Accordingly, the problem of the deformation of a layered plate for which $a \sim 2$ reduces in the zeroth approximation to a system of $2(n+m)+1$ equations in the $2(n+m)+1$ unknown functions

$$v_{\alpha 0}^{(2k,0)},\ v_{\beta 0}^{(2k,0)}\ (k = 1, 2, \ldots, n; \quad -1, -2, \ldots -m)\ v_{\gamma 0}^{(0)}$$

These equations have the form

$$- e_{2k}\,(\zeta_{2k} - \zeta_{2k-1})\, L(v_{\alpha 0}^{(2k,\,0)}, v_{\beta 0}^{(2k,\,0)}; \nu_{2k}) + \frac{e_{2k}\,(\zeta_{2k}^{2} - \zeta_{2k-1}^{2})}{4\,(1-\nu_{2k}^{2})}\,\frac{1}{H_\alpha}\,\frac{\partial}{\partial \xi}\,\nabla v_{\gamma 0}^{(2k,0)} =$$

$$= \frac{e_{2k+1}}{2(1+\nu_{2k+1})}\left[\frac{1}{\zeta_{2k+1}-\zeta_{2k}}\left(v_{\alpha 0}^{(2k+2,\,0)} - v_{\alpha 0}^{(2k,\,0)}\right) - \right.$$
$$\left. - \frac{1}{\zeta_{2k-1}-\zeta_{2k-2}}\left(v_{\alpha 0}^{(2k,\,0)} - v_{\alpha 0}^{(2k-2,\,0)}\right)\right] \qquad (\alpha\beta) \tag{6.5}$$

$$\sum{}^{*} (\zeta_{2k}^2 - \zeta_{2k-1}^2)\frac{e_{2k}}{2(1-\nu_{2k}^2)}\nabla\left\{\frac{1}{H_\alpha H_\beta}\left[\frac{\partial}{\partial\xi}(H_\beta v_{\alpha 0}^{(2k,\,0)}) + \frac{\partial}{\partial\eta}(H_\alpha v_{\beta 0}^{(2k,\,0)})\right]\right\} -$$
$$- Q_3\nabla\nabla v_{\gamma 0}^{(0)} = - \frac{1}{E_2\varepsilon^{\varkappa+4}}\left\{\tau_\gamma^+ - \tau_\gamma^- + \varepsilon\frac{1}{H_\alpha H_\beta}\left[\frac{\partial H_\beta(\zeta_{2n}\tau_\alpha^+ - \zeta_{-2m}\tau_\alpha^-)}{\partial\xi} + \right.\right.$$
$$\left.\left. + \frac{\partial H_\alpha(\zeta_{2n}\tau_\beta^+ - \zeta_{-2m}\tau_\beta^-)}{\partial\eta}\right]\right\} \tag{6.6}$$

For a symmetrically constructed, three-layered (sandwich) plate, Eqs. (6.5) and (6.6) are a system of three equations in the three unknowns $v_{\alpha 0}^{(2,0)}$, $v_{\beta 0}^{(2,0)}$, and $v_{\gamma 0}^{(0)}$.

Let us examine a layered plate for which $a > 2$. When we investigate the possibility of satisfying the boundary conditions and the conditions of bonding of the layers, we arrive at the conclusion that in the zeroth approximation the loaded layers take the entire surface load. This means that for $a > 2$ a layered plate ceases to act as a unit.

In the present paper only the internal state of stress has been studied. Therefore, the investigation which has been presented makes it possible to refine the differential equations of the internal problem for layered plates. However, in addition to the refinement of the differential equations, it is also necessary to carry through a refinement of the boundary conditions (for a homogeneous plate, see [6]). This is connected with the investigation of the states of stress corresponding to the edge effects.

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